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When is a symmetric body-bar structure isostatic?

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ABSTRACT

Body-bar frameworks provide a special class of frameworks which are well understood generically, with a full combinatorial theory for rigidity. Given a symmetric body-bar framework, this paper exploits group representation theory to provide necessary conditions for rigidity in the form of very simply stated restrictions on the numbers of those structural components that are unshifted by the symmetry operations of the framework. We give some initial results, and conjectures, for when these conditions are also sufficient for rigidity.

1. Introduction

This paper describes the conditions that symmetry places on body-bar frameworks that are isostatic, i.e., both statically and kinematically determinate, thus extending the work on bar and joint frameworks that was described in Connelly et al. (2009).

Body-bar frameworks consist of rigid bodies in a d-dimensional space that are connected together by rigid bars, each of which provides a length constraint between two joints which lie on different bodies. Body-bar frameworks provide a useful way of describing many structures and mechanisms. In particular, they avoid difficulties that occur using combinatorial algorithms to detect mechanisms and states of self-stress for bar and joint frameworks in 3D (Whiteley, 1996), where the 'double banana' (see, e.g., Fowler and Guest, 2002) provides a classic counter-example to the existence of a straightforward extension to 3D of the Laman (1970) characterisation of isostatic 2D bar and joint frameworks. Body-bar frameworks hold the promise of a systematic theory of rigidity which exhibits all the key combinatorial properties, theorems and algorithms of the well understood plane bar and joint structures (Tay, 1984; Whiteley, 1988; White and Whiteley, 1987). These good combinatorial properties are the reason that body-bar frameworks form the underlying model used in calculations regarding the flexibility of biomolecules (see e.g., Hespenheide et al., 2004).

A number of 'classical' linkages and robotic mechanisms have the structure of a body-bar framework. One simple example is the Stewart platform, which is two bodies joined by six bars (Fichter et al., 2009). The platform is manipulated by changing the length of the six bars (pistons). A key concern are the singular positions, where the structure both becomes dependent (has a static self-stress) and loses access to one of the original 6 degrees of freedom (Fichter, 1986). Fig. 1 shows examples of Stewart platforms where the actuating bars have been given a fixed length, so that they become rigid bars.

Body-bar frameworks may often be generated in a symmetric configuration, and this paper examines the impact of symmetry on the rigidity of the framework. The paper extends the prior work on necessary conditions imposed on bar and joint frameworks by various symmetry groups to provide necessary conditions on body-bar frameworks to remain isostatic. Further, the good combinatorial properties of body-bar frameworks raises the promise of converting these necessary conditions into necessary and sufficient conditions for frameworks with symmetry.

2. Background

2.1. Scalar counting rule

A d-dimensional body-bar framework consists of a set of b full-dimensional rigid bodies in $\mathbb{R}^d$ which are connected by e rigid bars. The bodies each move, preserving the distance between any two points that are connected by a bar. The underlying combinatorial structure for a body-bar framework in d-space is a multigraph $G=(B,E)$ which allows up to $\binom{d+1}{2}$ edges (forming a set E) between any pair of 'vertices' (forming a set of bodies, B). The upper bound on the number of bars is motivated by the fact that

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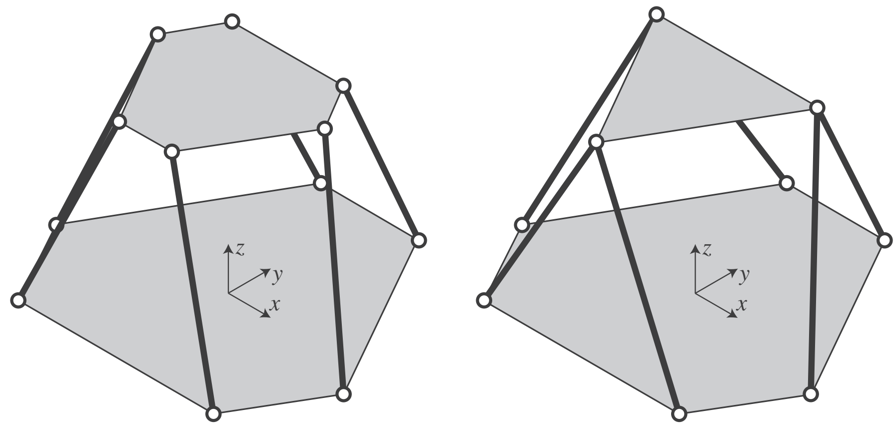

(a) (b)

Fig. 1. A Stewart platform is a simple example of a body-bar framework, which can become symmetric. In (a) all joints are distinct, in (b) some of the joints are identified. We will focus on structures of type (a).

the space of infinitesimal motions of a full-dimensional rigid body in d-space (such as a rigid bar and joint framework whose joints span all of $\mathbb{R}^d$) has dimension $\binom{d+1}{2}$. So, in order to join two rigid bodies in $\mathbb{R}^d$ in such a way that the resulting structure is again rigid, one needs $\binom{d+1}{2}$ properly placed bars, and additional bars will give a local overconstraint between the two bodies.

The configuration p of a d-dimensional body-bar framework $G(p)$ defines the positions of all the end-points of the bars of $G(p)$ in $\mathbb{R}^d$ (i.e., the attachment points of the bars on the bodies). We will restrict our configurations to realisations in which all the attachment points on a particular body are distinct, e.g., the system shown in Fig. 1a, and not that shown in Fig. 1b. Further, we only consider body-bar frameworks with injective configurations in this paper, and hence we do not allow attachment points to coincide at all. A number of subtle difficulties can occur in applying techniques from group representation theory to the analysis of body-bar frameworks with non-injective configurations. A detailed discussion of these difficulties can be found in Schulze (2010a), with further discussion in Section 6.1.

For an arbitrary dimension d, the following result has been proven by Tay in 1984 (see also White and Whiteley, 1987).

Theorem 1 (Tay, 1984). *For a generic body-bar configuration in* $\mathbb{R}^d$, *p, the body-bar framework G(p) is isostatic if and only if G = (B, E) satisfies the conditions:*

(i) $e = \binom{d+1}{2}b - \binom{d+1}{2}$;

(ii) *for any non-empty set of bodies* B^*, *which induce just the bars in* E^*, *with* $|B^*| = b^*$ *and* $|E^*| = e^*$, $e^* \leqslant \binom{d+1}{2}b^* - \binom{d+1}{2}$.

Equivalently, the body-bar framework G(p) is isostatic in d-space if and only if G = (B, E) is partitioned into $\binom{d+1}{2}$ *spanning trees.*

A simple counting rule can be developed from Theorem 1 for possibly non-generic frameworks (i.e., where the bodies and bars may not lie in a completely general position) by considering the linear algebra of an equilibrium or rigidity matrix (as described, for example, in Guest and Pellegrino (1994)), or can be derived as a special case of mobility counting, see Guest and Fowler (2005). For a system with an m-dimensional space of internal infinitesimal mechanisms, and an s-dimensional space of self-stresses, the counting rule is

$$m - s = \binom{d+1}{2}(b-1) - e. \quad (1)$$

Eq. (1) gives a simple counting condition for the determinacy of a d-dimensional body-bar framework, in terms of the number of 'vertices' (bodies), b, and the number of 'edges' (bars), e, of the structure. The number $m - s$ on the left hand side of Eq. (1) expresses the net freedom of the structure as the difference between the dimension of the space of infinitesimal internal mechanisms and the dimension of the space of self-stresses. A statically determinate structure has $s = 0$; a kinematically determinate structure has $m = 0$; isostatic structures have $s = m = 0$.

Throughout this paper we will slightly abuse notation by denoting the space of internal infinitesimal mechanisms and the space of self-stresses by the same symbols, m and s, as their respective dimensions.

2.2. Symmetry-extended counting rule

To formalize the notion of a symmetric body-bar framework $G(p)$, we consider the bar and joint framework $\overline{G}(p)$ which is obtained by replacing each body of $G(p)$ with the bar and joint realisation of the complete graph on the set of attachment points on the body. We define a *symmetry operation* of a body-bar framework $G(p)$ in $\mathbb{R}^d$ as an isometry R of $\mathbb{R}^d$ such that for some graph automorphism $\alpha \in \mathrm{Aut}(\overline{G})$, we have

$$R(p(v)) = p(\alpha(v)) \quad \text{for all } v \in V(\overline{G}).$$

The *symmetry element* corresponding to R is the affine subspace of points in $\mathbb{R}^d$ that are fixed by R (see Fig. 2, for example). The set of all symmetry operations of a body-bar framework $G(p)$ forms a group under composition, called the *point group* of $G(p)$.

Note that the symmetry operations in the point group $\mathcal{G}$ of a body-bar framework $G(p)$ induce permutations of both the bodies and the bars of $G(p)$. These permutations in turn give rise to two 'natural' group representations of $\mathcal{G}$: the 'internal' representation which describes how the bars are being permuted by each symmetry operation in $\mathcal{G}$, and the 'external' representation which describes how the bodies are being permuted and how the coordinate system for each body is effected by each symmetry operation in $\mathcal{G}$. These definitions of the internal and external representation are completely analogous to the definitions of the internal and external representation introduced in Kangwai and Guest (2000) and Fowler and Guest (2000) to establish the symmetry-extended version of Maxwell's rule for bar and joint frameworks (see also Schulze, in press-b, for further details). Using the basic

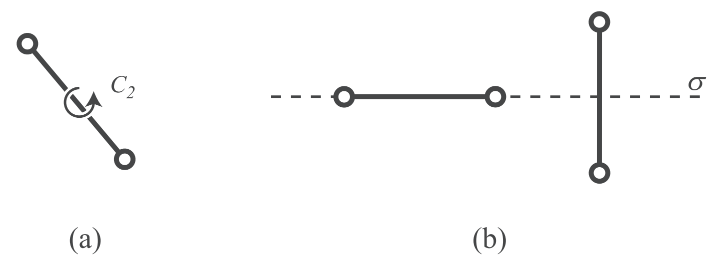

Fig. 2. Possible placement of a bar with respect to a symmetry element in two dimensions, such that it is unshifted by the associated symmetry operation: (a) C_2 centre of rotation; (b) mirror line.

techniques from group representation theory given in Fowler and Guest (2000) and Schulze (in press-b), we can refine the scalar counting rule in Eq. (1) to take the following 'symmetry-extended' form:

$$\Gamma(m) - \Gamma(s) = [\Gamma_T + \Gamma_R] \times [\Gamma(b) - \Gamma_0] - \Gamma(e). \qquad (2)$$

This could also be derived as a special case of the symmetry-adapted mobility rule given in Guest and Fowler (2005).

In Eq. (2), each Γ is known in mathematical group theory as a *character* (James and Liebeck, 2001), and in applied group theory as a *representation* of $\mathcal{G}$ (Bishop, 1973). For any set of objects q, $\Gamma(q)$ can be considered as a vector, or ordered set, of the traces of the transformation matrices $\mathbf{D}_q(R)$ that describe the transformation of q under each symmetry operation R that lies in $\mathcal{G}$. In this way, (2) may be considered as a set of equations, one for each conjugacy class of symmetry operations in $\mathcal{G}$. Alternatively, and equivalently, each $\Gamma(q)$ can be written as the sum of irreducible representations/characters of $\mathcal{G}$ (Bishop, 1973). In (2) the various sets q are sets of bodies b, bars e, mechanisms m and states of self-stress s; Γ_0 is the trivial representation which takes the value of one for all group elements, and Γ_T and Γ_R are the representations of translations and rotations in d-space, respectively (see also Schulze, 2009b).

In 3-space, Eq. (2) becomes

$$\text{3D}: \quad \Gamma(m) - \Gamma(s) = [\Gamma_{x,y,z} + \Gamma_{R_x,R_y,R_z}] \times [\Gamma(b) - \Gamma_0] - \Gamma(e), \qquad (3)$$

where $\Gamma_{x,y,z}$ is the representation of translations along the three Cartesian directions and Γ_{R_x,R_y,R_z} is the representation of rotations about the three Cartesian directions. In the 3-dimensional case, calculations using (3) can be completed by standard manipulations of the character table of the group (Atkins et al., 1970; Bishop, 1973; Altmann and Herzig, 1994).

Analogously, for 2-dimensional body-bar frameworks (assumed to lie in the xy-plane), Eq. (2) becomes

$$\text{2D}: \quad \Gamma(m) - \Gamma(s) = \left[\Gamma_{x,y} + \Gamma_{R_z}\right] \times [\Gamma(b) - \Gamma_0] - \Gamma(e). \qquad (4)$$

Note that Eq. (4) is obtained from Eq. (3) by replacing $\Gamma_{x,y,z}$ with $\Gamma_{x,y}$ and Γ_{R_x,R_y,R_z} with Γ_{R_z}, as appropriate to the reduced set of rigid-body motions.

In the context of the present paper, we are interested in isostatic systems, which have $m = s = 0$, and hence obey the symmetry condition $\Gamma(m) = \Gamma(s) = 0$. In fact, the symmetric version of Tay's Eqs. (2)–(4) gives the necessary but not sufficient condition $\Gamma(m) - \Gamma(s) = 0$, as it cannot detect the presence of paired equisymmetric mechanisms and states of self stress.

The symmetry-extended Tay equation corresponds to a set of k scalar equations, where k is the number of irreducible representations of $\mathcal{G}$ (the number of rows in the character table), or equivalently the number of conjugacy classes of $\mathcal{G}$ (the number of columns in the character table). The former view has been used in Fowler and Guest (2000) and Schulze (in press-b); the latter view has recently been used in Connelly et al. (2009) to formulate the additional necessary conditions for a symmetric bar and joint framework to be isostatic in terms of simply stated restrictions on the numbers of joints and bars that are unshifted by various symmetry operations of the framework. In this paper, we again use the latter view to establish analogous restrictions on isostatic symmetric body-bar frameworks.

A related analysis for bar and joint frameworks, which could also be extended to body-bar frameworks, is given by Owen and Power (2008).

3. Two-dimensional isostatic body-bar frameworks

In this section we treat the two-dimensional case: bars, joints, and bodies, and their associated displacements are all confined to the plane. (Note that frameworks that are isostatic in the plane may have out-of-plane mechanisms when considered in 3-space.) We use the Schoenflies notation for symmetry operations (see, e.g., Altmann and Herzig, 1994). The relevant symmetry operations are: the identity (E), rotation by $2\pi/n$ about a point (C_n), and reflection in a line (σ). The possible groups are the groups $\mathcal{C}_n$ and $\mathcal{C}_{nv}$ for all natural numbers n. $\mathcal{C}_n$ is the cyclic group generated by C_n, and $\mathcal{C}_{nv}$ is generated by a $\{C_n, \sigma\}$ pair. The group $\mathcal{C}_{1v}$ is usually called $\mathcal{C}_s$.

All two-dimensional cases can be treated in a single calculation, as shown in Table 1. Characters are calculated for four operations: we distinguish C_2 from the C_n operation with $n > 2$. Each line in the table represents a stage in the evaluation of (4). Similar tabular calculations are found in Fowler and Guest (2000) and subsequent papers such as Connelly et al. (2009).

To treat all two-dimensional cases in a single calculation, we need a notation that keeps track of the fate of structural components under the various operations, which in turn depends on how the bodies and bars are placed with respect to the symmetry elements. A key concept is whether a component is shifted or unshifted by a symmetry operation: loosely, a component (body, bar) is unshifted if it is not moved (but may be reoriented) by a symmetry operation. More precisely, given a body-bar framework with point group $\mathcal{G}$, we say that a body is *unshifted* by a symmetry operation R in $\mathcal{G}$ if it is fixed by the permutation of the bodies induced by R, i.e., if each attachment point on the body is mapped

Table 1
Calculations of representations for the 2D symmetry-extended Tay Eq. (4) for body-bar frameworks in the plane.

	E	σ	C_2	$C_{n>2}(\phi)$
$\Gamma(b)$	b	b_σ	b_2	b_n
$-\Gamma_0$	-1	-1	-1	-1
$=[\Gamma(b) - \Gamma_0]$	$b-1$	$b_\sigma - 1$	$b_2 - 1$	$b_n - 1$
$\times[\Gamma_{xy} + \Gamma_{R_z}]$	3	-1	-1	$2\cos\phi + 1$
$= [\Gamma(b) - \Gamma_0] \times [\Gamma_{xy} + \Gamma_{R_z}]$	$3(b-1)$	$-b_\sigma + 1$	$-b_2 + 1$	$(b_n - 1)(2\cos\phi + 1)$
$-\Gamma(e)$	$-e$	$-e_\sigma$	$-e_2$	0
$=\Gamma(m) - \Gamma(s)$	$3b - e - 3$	$-b_\sigma - e_\sigma + 1$	$-b_2 - e_2 + 1$	$(b_n - 1)(2\cos\phi - 1)$

to a (possibly different) attachment point on the *same* body; similarly, a bar is *unshifted* by R if either $R(p(v)) = p(v)$ and $R(p(w)) = p(w)$ or $R(p(v)) = p(w)$ and $R(p(w)) = p(v)$, where $p(v)$ and $p(w)$ are the endpoints of the bar (see also Figs. 2–6). The notation used in Table 1 is as follows:

- b total number of bodies;
- b_n number of bodies which are unshifted by a given n-fold rotational symmetry operation $C_{n\geqslant 2}$;
- b_σ number of bodies unshifted by a given reflection σ;
- e total number of bars;
- e_2 number of bars left unshifted by a C_2 operation (see Fig. 2a and note that C_n with $n > 2$ shifts all bars);
- e_σ number of bars unshifted by a given reflection σ (see Fig. 2b: the unshifted bar may lie in, or perpendicular to, the mirror line).

Each of the counts refers to a particular symmetry element and any structural component may therefore contribute to one or more count, for instance, a body counted in b_n also contributes to b_σ if it lies on a rotation axis and a reflection line.

From Table 1, the symmetry treatment of the 2D body-bar equation reduces to scalar equations of four types. If $\Gamma(m) - \Gamma(s) = 0$, then

$$E: \quad 3b - e = 3, \tag{5}$$

$$\sigma: \quad b_\sigma + e_\sigma = 1, \tag{6}$$

$$C_2: \quad b_2 + e_2 = 1, \tag{7}$$

$$C_{n>2}: \ (b_n - 1)(2\cos\phi + 1) = 0, \tag{8}$$

where a given equation applies when the corresponding symmetry operation is present in $\mathcal{G}$.

Some observations on 2D isostatic body-bar frameworks, arising from this set of equations are:

(i) Trivially, all 2D body-bar frameworks have the identity element and (5) simply restates the scalar Tay rule (1) for $m - s = 0$.

(ii) Presence of a mirror line implies, by (6), that either $b_\sigma = 1$, $e_\sigma = 0$ or $b_\sigma = 0$, $e_\sigma = 1$. Note, however, that for the second case, the bar must lie perpendicular to the mirror: if the bar lay on the mirror, the two end bodies must also have the symmetry of the mirror, implying $b_\sigma \geqslant 2$.

(iii) Presence of a C_2 element imposes limitations on the placement of bodies and bars. As both b_2 and e_2 must be non-negative integers, (7) has two solutions: $b_2 = 1$, $e_2 = 0$ or $b_2 = 0$,

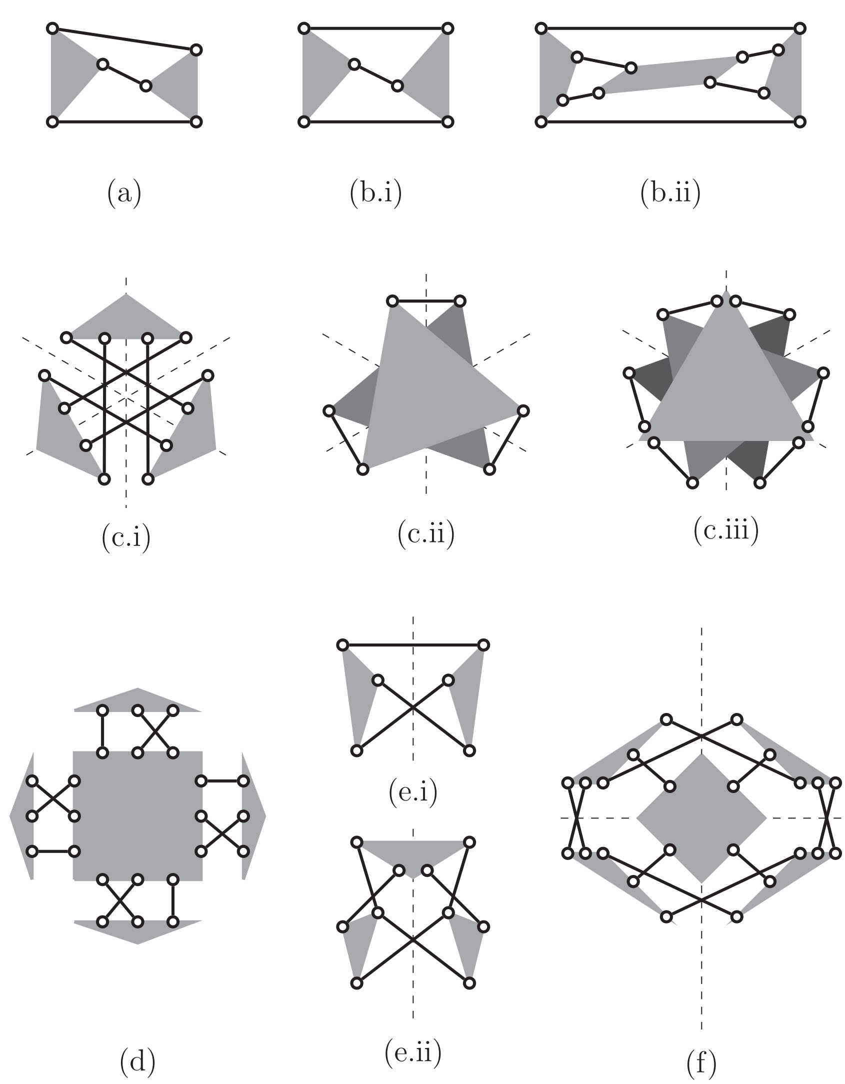

Fig. 3. Examples, for various point groups, of small 2D isostatic body-bar frameworks: (a) $\mathcal{C}_1$; (b) $\mathcal{C}_2$ with (i) $b_2 = 0$, $e_2 = 1$, and (ii) $b_2 = 1$, $e_2 = 0$; (c) $\mathcal{C}_{3v}$ with (i) $b_3 = 0$, (ii) $b_3 = 2$, and (iii) $b_3 = 3$; note that one can easily obtain isostatic body-bar frameworks with point group symmetry $\mathcal{C}_3$ from the examples in (c) by appropriately perturbing the positions of the joints; (d) $\mathcal{C}_4$; note that this example can easily be generalized to obtain examples for any $\mathcal{C}_n$, $n \geqslant 2$; (e) $\mathcal{C}_s$ with (i) $b_\sigma = 0$, $e_\sigma = 1$, and (ii) $b_\sigma = 1$, $e_\sigma = 0$; (f) $\mathcal{C}_{2v}$; this example can again easily be generalized to obtain isostatic body-bar frameworks with point group symmetry $\mathcal{C}_{nv}$, for any $n \geqslant 2$.

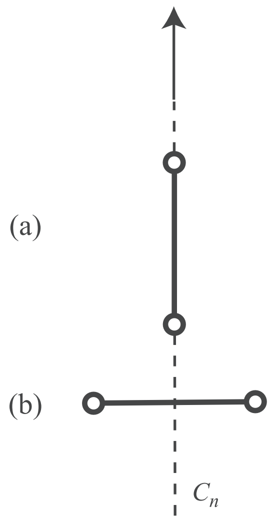

Fig. 4. Possible placement of a bar unshifted by a proper rotation about an axis: (a) for any $C_{n\geqslant 2}$; (b) for C_2 alone.

$e_2 = 1$. In other words, an isostatic 2D body-bar framework with a C_2 symmetry has either exactly one body unshifted and no bar unshifted, or exactly one bar centered on the point of rotation (unshifted) and no body unshifted.

(iv) For $C_{n>2}$, Eq. (8) with $\phi = 2\pi/n$ implies

$$(b_n - 1)\left(2\cos\left(\frac{2\pi}{n}\right) + 1\right) = 0 \quad (9)$$

and hence for all n, $b_n = 1$ is a possible solution. Alternatively, we could have $\cos(2\pi/n) = -1/2$, implying that for $n = 3$ there is no restriction on b_3, the number of bodies unshifted by a 3-fold rotation.

In summary, a 2D isostatic body-bar framework may have symmetry operations drawn from the list $\{E, C_2, C_3, C_n(n > 3), \sigma\}$, and hence the possible symmetry groups $\mathcal{G}$ are infinite in number: $\mathcal{C}_1$, $\mathcal{C}_n$, $\mathcal{C}_s$, $\mathcal{C}_{nv}$. Group by group, the conditions necessary for a 2D body-bar framework to be isostatic are then as follows:

$\mathcal{C}_1$: $e = 3b - 3$.

$\mathcal{C}_2$: $e = 3b - 3$ with: (i) $b_2 = 1$, $e_2 = 0$ and all other bodies and all edges occurring in pairs, implying b odd and e even; or (ii) $b_2 = 0$, $e_2 = 1$ and all bodies and all other edges occurring in pairs, implying b even and e odd (see Fig. 3b).

$\mathcal{C}_3$: $e = 3b - 3$ with b_3 arbitrary, and all bars occurring in sets of 3 (see Fig. 3c).

$\mathcal{C}_n$, $n > 3$: $e = 3b - 3$ with $b_n = 1$, and hence all but one body occurring in sets of n. If $n = 2m$, then the induced C_2 tells us there is no centered bar and all bars occur in sets of n (see Fig. 3d). If n is odd, then there can be centered bars, but they occur in sets of n, as they are shifted.

$\mathcal{C}_s$: $e = 3b - 3$ with: (i) $b_\sigma = 1$, $e_\sigma = 0$; or (ii) $b_\sigma = 0$, $e_\sigma = 1$. Either one body and no bar or one bar and no body is unshifted by the mirror, and all other bodies and bars occur in sets of two (see Fig. 3e).

$\mathcal{C}_{2v}$: $e = 3b - 3$ with $b_2 = b_\sigma = 1$ and $e_2 = e_\sigma = 0$. There is a central body with full $\mathcal{C}_{2v}$ symmetry and no bars are either centered on the axis, or unshifted by a mirror (see Fig. 3f). All bars occur in sets of 4 and all bodies beyond the centered body are off mirrors, and hence also occur in sets of 4. Note that $e_2 = 1$ is not possible, as this bar must lie on one of the mirrors, implying that the bodies at its ends also lie on the mirror, which would violate the required $b_\sigma = 0$.

$\mathcal{C}_{3v}$: (i) $e = 3b - 3$ with b_3 arbitrary (Fig. 3c). With the mirrors, we can either have $b_\sigma = 1$ and $e_\sigma = 0$, or have $b_\sigma = 0$ and $e_\sigma = 1$, where for each of the three mirror lines, the bar that is unshifted by the mirror is perpendicular to, and centered on, the mirror.

$\mathcal{C}_{nv}$, $n > 3$: $e = 3b - 3$ with $b_n = 1$ and $e_\sigma = 0$. There is a central body with full $\mathcal{C}_{nv}$ symmetry. There can be bars centered on the rotation centre if n is odd (they are shifted), but there cannot be any bars centered on the rotation centre if n is even (see Fig. 3f).

We consider whether these conditions are also sufficient in Section 5.1.

Note that an isostatic body-bar framework can be constructed for any given point group in 2D. Examples of small 2D isostatic body-bar frameworks for various point groups are depicted in Fig. 3.

4. Three-dimensional isostatic body-bar frameworks

The families of possible point groups of 3D objects are: the icosahedral $\mathcal{I}, \mathcal{I}_h$; the cubic $\mathcal{T}, \mathcal{T}_h, \mathcal{T}_d, \mathcal{O}, \mathcal{O}_h$; the axial $\mathcal{C}_n, \mathcal{C}_{nh}, \mathcal{C}_{nv}$; the dihedral $\mathcal{D}_n, \mathcal{D}_{nh}, \mathcal{D}_{nd}$; the cyclic $\mathcal{S}_{2n}$; and the trivial $\mathcal{C}_s, \mathcal{C}_i, \mathcal{C}_1$ (Atkins et al., 1970). The relevant symmetry operations are: proper rotation by $2\pi/n$ about an axis, C_n, and improper rotation, S_n (C_n followed by reflection in a plane perpendicular to the axis). By convention, the identity $E \equiv C_1$, inversion $i \equiv S_2$, and reflections $\sigma \equiv S_1$ are treated separately.

The calculation of characters for the 3D symmetry-extended Tay Eq. (3) is shown in Table 2. Characters are calculated for six operations. For proper rotations, we distinguish E and C_2 from the C_n operations with $n > 2$. For improper rotations, we distinguish σ and i from the $S_{n>2}$ operations.

The notation used in Table 2 is

b total number of bodies;

b_n number of bodies which are unshifted by a given n-fold rotational symmetry operation $C_{n\geqslant 2}$;

b_c number of bodies unshifted by the inversion i or the improper rotation $S_{n>2}$; each such body is centered on the unique central point;

b_σ number of bodies unshifted by a given reflection σ;

e total number of bars;

e_n number of bars unshifted by a $C_{n>2}$ rotation: note that each such bar must lie along the axis of the rotation (see Fig. 4a);

e_2 number of bars unshifted by a given C_2 rotation: such bars must lie either along, or perpendicular to and centered on, the axis (see Fig. 4a and b);

e_{nc} number of bars unshifted by the improper rotation $S_{n>2}$: note that such bars must lie along the axis of the rotation, and be centered on the central point of the group (see Fig. 5a);

e_i number of bars unshifted by the inversion i: note that the centre of the bar must lie at the central point of the group, but no particular orientation is implied (see Fig. 5b);

e_σ number of bars unshifted by a given reflection σ: an unshifted bar may lie on the mirror or perpendicular to and centered on the mirror (Fig. 6a and b).

Again, each of the counts refers to a particular symmetry element, and so, for instance a body counted in b_c also contributes to b, and may contribute to b_n and b_σ if it has these symmetries.

From Table 2, the symmetry treatment of the 3D Tay equation reduces to scalar equations of six types. If $\Gamma(m) - \Gamma(s) = 0$, then

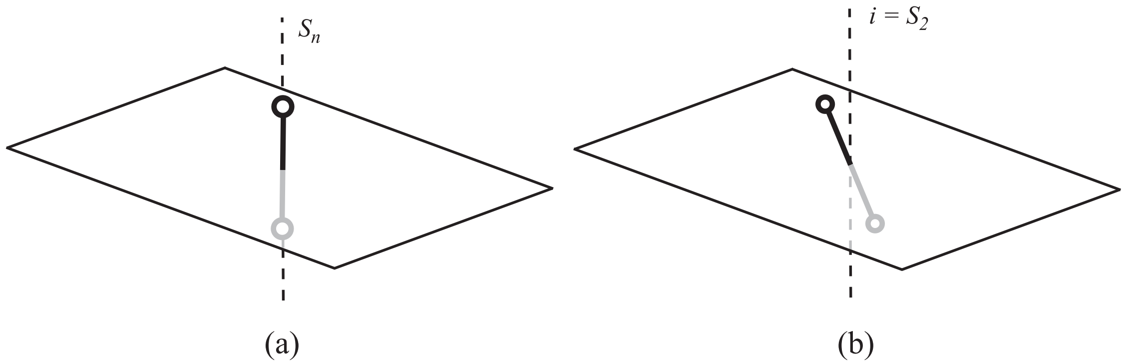

Fig. 5. Possible placement of a bar unshifted by an improper rotation about an axis: (a) for any $S_{n\geqslant 2}$; (b) for $i = S_2$.

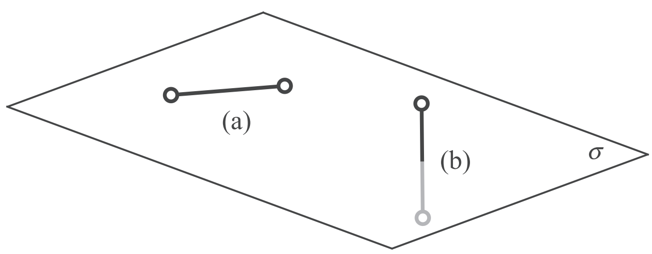

Fig. 6. Possible placement of a bar unshifted by a reflection in a plane: (a) lying in the plane; (b) lying perpendicular to the plane.

$$E: \quad 6b - 6 = e, \tag{10}$$
$$\sigma: \quad e_\sigma = 0, \tag{11}$$
$$i: \quad e_i = 0, \tag{12}$$
$$S_{n>2}: \quad e_{nc} = 0, \tag{13}$$
$$C_2: \quad 2b_2 + e_2 = 2, \tag{14}$$
$$C_{n>2}: \quad (b_n - 1)(4\cos\phi + 2) = e_n, \tag{15}$$

where a given equation applies when the corresponding symmetry operation is present in $\mathcal{G}$.

Some observations on 3D isostatic body-bar frameworks, arising from the above, are:

(i) From (10), the body-bar framework must satisfy the scalar Tay rule (1) with $m - s = 0$: $6(b - 1) = e$.
(ii) From (11), each mirror σ that is present contains an arbitrary number of bodies that are unshifted by σ, but there are no bars in the mirror or bars perpendicular to and centered on the mirror.
(iii) From (12), a centro-symmetric body-bar framework has no bar centered at the inversion centre, and the number of centrally symmetric bodies is arbitrary.
(iv) From (13), the presence of an improper rotation $S_{n>2}$ implies that no bar lies on the improper rotation axis, and the number of bodies unshifted by $S_{n>2}$ is arbitrary.
(v) For a C_2 axis, (14) has solutions

$$(b_2, e_2) = (1, 0) \text{ or } (0, 2).$$

The count e_2 refers to both bars that lie along, and those that lie perpendicular to the axis. However, if a bar were to lie along the C_2 axis, the bodies at either end would contribute 2 to b_2, thus generating a contradiction to (14), so all bars included in e_2 must lie perpendicular to the axis.
(vi) Eq. (15) can be written, with $\phi = 2\pi/n$, as

$$(b_n - 1)\left(4\cos\left(\frac{2\pi}{n}\right) + 2\right) = e_n$$

with $n > 2$. The non-negative integer solution $b_n = 1$, $e_n = 0$, is possible for all n. For $n > 2$ the factor $(4\cos(2\pi/n) + 2)$ is rational at $n = 3$, 4, 6, but generates a further distinct solution only for $n = 3$:
$n = 3$

$$0(b_3 - 1) = e_3$$

and so here $e_3 = 0$, but b_3 is unrestricted.
$n = 4$

$$2(b_4 - 1) = e_4.$$

One possibility is $b_4 = 1$, which covers all the requirements, with $e_4 = e_2 = 0$. If we consider the option of $b_4 = b_2 = 0$, then we have $e_4 < 0$ which is impossible. If we consider $b_4 > 1$, then C_4 implies $C_4^2 = C_2$ about the same axis, and hence $b_4 = b_2 > 1$, which is also impossible. Thus we only have the one case $b_4 = 1$.
$n = 6$

$$4(b_6 - 1) = e_6.$$

C_6 implies $C_6^3 = C_2$ and $C_6^2 = C_3$ about the same axis, and hence $e_6 = e_3 = 0$, and $b_6 = b_3 = b_2 = 1$.

Thus e_n is 0 for any $n > 2$, and only in the case $n = 3$ may b_n depart from 1.

The above conditions do not exclude any point groups; however, for particular groups, some further interesting observations can be made.

Table 2
Calculations of representations for the 3D symmetry-extended Tay Eq. (3) for body-bar frameworks in 3-space.

	E	σ	i	$S_{n>2}$	C_2	$C_{n>2}(\phi)$
$\Gamma(b)$	b	b_σ	b_c	b_c	b_2	b_n
$-\Gamma_0$	-1	-1	-1	-1	-1	-1
$=\Gamma(b) - \Gamma_0$	$b - 1$	$b_\sigma - 1$	$b_c - 1$	$b_c - 1$	$b_2 - 1$	$b_n - 1$
$\times(\Gamma_{xyz} + \Gamma_{R_xR_yR_z})$	6	0	0	0	-2	$4\cos\phi + 2$
$= (\Gamma(b) - \Gamma_0) \times (\Gamma_{xyz} + \Gamma_{R_xR_yR_z})$	$6(b - 1)$	0	0	0	$-2(b_2 - 1)$	$(4\cos\phi + 2)(b_n - 1)$
$-\Gamma(e)$	$-e$	$-e_\sigma$	$-e_i$	$-e_{nc}$	$-e_2$	$-e_n$
$=\Gamma(m) - \Gamma(s)$	$6(b - 1) - e$	$-e_\sigma$	$-e_i$	$-e_{nc}$	$-2b_2 + 2 - e_2$	$(4\cos\phi + 2)(b_n - 1) - e_n$

(i) For $\mathcal{C}_{2v}$, there are no added constraints, but we observe that if $e_2 = 2$, then the two bars perpendicular to the axis are mirror images of each other and not in either mirror.

(ii) For $\mathcal{C}_{nv}$, $n \geqslant 3$, there are no added constraints. However, note that for $n > 3$, the body which is unshifted by C_n must have the full $\mathcal{C}_{nv}$ symmetry, for otherwise we have $b_n > 1$.

(iii) For $\mathcal{C}_{2h}$, we observe that if $b_2 = 1$ and $e_2 = 0$, then the body that is unshifted by C_2 must also be unshifted by the reflection σ whose mirror is perpendicular to the C_2 axis (for otherwise we have $b_2 > 1$), and hence also by the inversion i; this body is therefore centered on the point of inversion and has full $\mathcal{C}_{2h}$ symmetry. If $b_2 = 0$ and $e_2 = 2$, then the two bars perpendicular to the axis are mirror images of each other.

(iv) For $\mathcal{C}_{3h}$, there are no added constraints since b_3 is arbitrary.

(v) For $\mathcal{C}_{nh}$, $n > 3$, the body that is unshifted by C_n must also be unshifted by the reflection σ (whose mirror is perpendicular to the C_n axis), and hence also by the improper rotation S_n. So, this body is a central body with full $\mathcal{C}_{nh}$ symmetry.

(vi) For $\mathcal{D}_2$, we observe that if there exists a body that is unshifted by one of the 2-fold rotations, then this body must also be unshifted by the other two 2-fold rotations. This body must therefore be centered on the intersection point of the three 2-fold axes, with full $\mathcal{D}_2$ symmetry.

(vii) For $\mathcal{D}_3$, there are no added constraints.

(viii) For $\mathcal{D}_n$, $n > 3$, we observe that the body which is unshifted by C_n must also be unshifted by each of the 2-fold rotations in $\mathcal{D}_n$ (whose axes are perpendicular to the C_n axis). This body is therefore centered on the intersection point of the rotational axes and has full $\mathcal{D}_n$ symmetry. In particular, it follows that we must have $b_2 = 1$ and $e_2 = 0$ for each C_2.

(ix) For $\mathcal{D}_{2h}$, we observe that if $b_2 = 1$ for one of the 2-fold rotations, then this body must also be unshifted by all the other elements in the group, so that it is centered at the point of inversion and has full $\mathcal{D}_{2h}$ symmetry. Any other bodies unshifted by the reflection in $\mathcal{D}_{2h}$ will be off the C_2 axis.

(x) For $\mathcal{D}_{3h}$, there are no added constraints.

(xi) For $\mathcal{D}_{nh}$, $n > 3$, the body which is unshifted by C_n must also be unshifted by all the other elements of the group and is hence centered on the point of intersection of the rotational axes, with full $\mathcal{D}_{nh}$ symmetry. In particular, we must have $b_2 = 1$ and $e_2 = 0$ for each C_2. Any other bodies unshifted by the reflection in $\mathcal{D}_{nh}$ have to lie off the rotational axes.

(xii) For $\mathcal{D}_{2d}$ and $\mathcal{D}_{3d}$, we observe that if there exists a body that is unshifted by one of the 2-fold rotations, then this body must also be unshifted by all the other elements in the group, so that it is a central body with the full symmetry of the group. For $\mathcal{D}_{2d}$, any other bodies unshifted by the reflection will be off the C_2 axis.

(xiii) For $\mathcal{D}_{nd}$, $n > 3$, the body which is unshifted by C_n must also be unshifted by all the other elements in the group, so that it is a central body with full $\mathcal{D}_{nd}$ symmetry. In particular, we must have $b_2 = 1$ and $e_2 = 0$ for each C_2. Any other bodies unshifted by one of the reflections in $\mathcal{D}_{nd}$ will be off the C_n axis.

(xiv) For $\mathcal{S}_4$, we observe that if there exists a body that is unshifted by the 2-fold rotation, then this body must be a central body with full $\mathcal{S}_4$ symmetry. Alternatively, if $b_2 = 0$ and $e_2 = 2$, then these two bars will be a pair of 'opposite' bars perpendicular to the C_2 axis.

(xv) For $\mathcal{S}_6$, there are no added constraints since there are no requirements on b_3 for the 3-fold axis.

(xvi) For $\mathcal{S}_{2n}$, $n > 3$, the body which is unshifted by C_n must also be unshifted by all the other elements in the group, so that it is a central body with full $\mathcal{S}_{2n}$ symmetry. In particular, if there exists a C_2 in $\mathcal{S}_{2n}$, we must have $b_2 = 1$ and $e_2 = 0$.

(xvii) For a body-bar framework with the rotational symmetries of a tetrahedron ($\mathcal{T}$), we observe that if we have $b_2 = 1$, then this must be a central body with full $\mathcal{T}$ symmetry. Alternatively, we have $b_2 = 0$ and $e_2 = 2$. For each of the C_2 rotations, these two bars would be a pair of 'opposite' bars perpendicular to the axis.

(xviii) For $\mathcal{T}_h$ and $\mathcal{T}_d$, there must exist a central body which has the full symmetry of the group. In particular, we must have $b_2 = 1$ and $e_2 = 0$ for each C_2. Any other bodies unshifted by a reflection will be off the C_2 axes.

(xix) For a body-bar framework with octahedral ($\mathcal{O}$ or $\mathcal{O}_h$) symmetry, the requirement that $b_4 = 1$ for each 4-fold axis implies that the structure must have one body centered where the axes meet, with the respective octahedral symmetry. In particular, we must have $b_2 = 1$ and $e_2 = 0$ for each C_2. For $\mathcal{O}_h$, any other bodies unshifted by a reflection will be off the C_2 and C_4 axes.

(xx) For a body-bar framework with icosahedral ($\mathcal{I}$ or $\mathcal{I}_h$) symmetry, the requirement that $b_5 = 1$ for each 5-fold axis implies that the structure must include a central body with the respective icosahedral symmetry. In particular, we must have $b_2 = 1$ and $e_2 = 0$ for each C_2. For $\mathcal{I}_h$, any other bodies unshifted by a reflection will be off the C_2 and C_5 axes.

As an example, we consider two problematic positions of the Stewart platform, as shown in Fig. 7. The 'standard' starting point, with 3-fold rotation on an axis through the two bodies satisfies the conditions above, and is indeed isostatic. However, if there is a 6-fold rotation axis (Fig. 7a), the condition $b_6 = 1$ is violated, and the configuration is singular, with both a stress and an infinitesimal motion which is not accessible to the control of the pistons. Similarly, if we have a mirror on two of the bars (and therefore a mirror symmetry of the two bodies) the configuration is singular (Fig. 7b). An explicit tabular calculation of characters for every symmetry operation for both structures is given in Table 3.

For bar and joint frameworks in 3D we had additional necessary conditions related to potential 'flatness' of sets of vertices and edges (Connelly et al., 2009). However, as long as our structures are 'combinatorially generic'—the ends of distinct bars are distinct points—then these examples cannot arise for symmetric body-bar frameworks. If, on the other hand, we build up a significant number of 'identifications' of end points (which implies coplanarity of bars) then there is a risk of some flatness requirements being imported with this specialization.

5. Sufficient conditions for isostatic body-bar realisations

A key goal of combinatorial characterizations for generic rigidity is to provide necessary and sufficient conditions, in the spirit of Laman's Theorem and Tay's Theorem (Theorem 1) for generic frameworks without symmetry.

For a body-bar framework with point-group symmetry $\mathcal{G}$ the previous sections have provided some necessary conditions for the realisation to be isostatic. These conditions included some over-all counts on bars and joints, along with sub-counts on special classes of bodies and bars (bars on mirrors or perpendicular to mirrors, bars centered on the axis of rotation, symmetric bodies on the

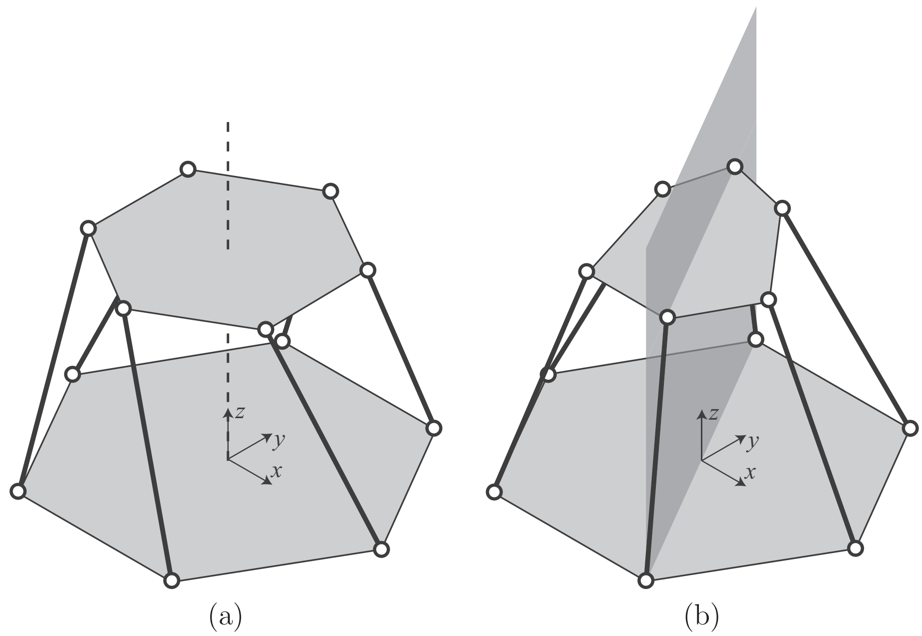

Fig. 7. Stewart platforms that are in singular positions due to the presence of symmetry: (a) a Stewart platform with $\mathcal{C}_6$ symmetry about the z-axis, shown dashed; (b) a Stewart platform with $\mathcal{C}_s$ symmetry in the $x = -y$ plane, shaded.

Table 3
Calculations of representations for the 3D symmetry-extended Tay Eq. (3) for the Stewart platform examples in Fig. 7a and b. As the final row of the table does not contain only zeros in either case, neither platform is isostatic for the particular symmetry given.

(a) $\mathcal{C}_6$	E	C_6	C_3	C_2	C_3^{-1}	C_6^{-1}
$\Gamma(b)$	2	2	2	2	2	2
$-\Gamma_0$	−1	−1	−1	−1	−1	−1
$=\Gamma(b) - \Gamma_0$	1	1	1	1	1	1
$\times(\Gamma_{xyz} + \Gamma_{R_xR_yR_z})$	6	4	0	−2	0	4
$= (\Gamma(b) - \Gamma_0) \times (\Gamma_{xyz} + \Gamma_{R_xR_yR_z})$	6	4	0	−2	0	4
$-\Gamma(e)$	−6	0	0	0	0	0
$=\Gamma(m) - \Gamma(s)$	0	4	0	−2	0	4

(b) $\mathcal{C}_s$	E	σ
$\Gamma(b)$	2	2
$-\Gamma_0$	−1	−1
$=\Gamma(b) - \Gamma_0$	1	1
$\times(\Gamma_{xyz} + \Gamma_{R_xR_yR_z})$	6	0
$= (\Gamma(b) - \Gamma_0) \times (\Gamma_{xyz} + \Gamma_{R_xR_yR_z})$	6	0
$-\Gamma(e)$	−6	−2
$=\Gamma(m) - \Gamma(s)$	0	−2

centre of rotation, etc.). Here, assuming that the framework is realized with the end-points of the bars (the attachments of bodies) in a configuration as generic as possible (subject to the symmetry conditions), we investigate whether these conditions are sufficient to guarantee that the framework is isostatic.

5.1. Sufficient conditions for 2D isostatic body-bar frameworks

The simplest case is the identity group ($\mathcal{C}_1$). For this basic situation, the key result is the 2D version of Tay's Theorem which can also be extracted from Laman's Theorem for bar and joint frameworks. In the following, we take the multigraph $G = (B,E)$ to define the connectivity of the body-bar framework, where B is the set of b bodies and E the set of e bars, and we take p to define the positions of all of the attachments in 2D. We recall the plane version of Tay's Theorem.

Theorem 2 (Tay, 1984). *For a generic body-bar configuration in 2D, p, the body-bar framework G(p) is isostatic if and only if G = (B,E) satisfies the conditions:*

(i) *$e = 3b - 3$;*
(ii) *for any non-empty set of bodies B^*, which induce just the bars in E^*, with $|B^*| = b^*$ and $|E^*| = e^*$, $e^* \leqslant 3b^* - 3$.*

Equivalently, the body-bar framework G(p) is isostatic in 2-space if and only if G = (B,E) is partitioned into three spanning trees.

Our goal is to extend these results to other symmetry groups. With the appropriate definition of 'generic' configurations for symmetry groups (Schulze, 2010a), we can anticipate that the necessary conditions identified in the previous sections for the corresponding group plus the condition identified in Theorem 2, which considers subgraphs that are not necessarily symmetric, will be sufficient.

For three of the plane symmetry groups, this has been confirmed. We use the previous notation for the point groups and the identification of special bodies and edges, and describe a configuration as 'generic with symmetry group $\mathcal{G}$' if, apart from conditions imposed by symmetry, the attachment points are in a generic position (the constraints imposed by the local site-symmetry may remove 0, 1 or 2 of the two basic freedoms of the point).

By embedding the body-bar framework as a bar and joint framework, with isostatic bar and joint bodies (of required symmetry) and applying the previous results for isostatic bar and joint symmetric bodies (Schulze, in press-a), the following cases can be verified.

Theorem 3. *If p is a plane configuration generic with symmetry group $\mathcal{G}$, and G(p) is a framework realized with these symmetries, then the following necessary conditions are also sufficient for G(p) to be isostatic:*

$e = 3b - 3$ and for any non-empty set of bars B^*, $e^* \leqslant 3b^* - 3$ and

(i) *for C_s: (a) $b_\sigma = 1$, $e_\sigma = 0$ or (b) $b_\sigma = 0$, $e_\sigma = 1$ (with all bars unshifted by σ perpendicular to the mirror);*
(ii) *for C_2: (a) $b_2 = 1$, $e_2 = 0$ or (b) $b_2 = 0$, $e_2 = 1$;*
(iii) *for C_3 : b_3 is arbitrary.*

There are also equivalent necessary and sufficient tree characterizations which apply to these groups, as translations from the results of Schulze (in press-a).

For the remaining groups, we have the corresponding conjectures. In some cases, these could not be generalizations of plane bar and joint framework results, since $\mathcal{C}_{n>3}$ and $\mathcal{C}_{nv:n>3}$ do not have isostatic bar and joint frameworks (Connelly et al., 2009).

Conjecture 1. *If p is a plane configuration generic with symmetry group $\mathcal{G}$, and G(p) is a framework realized with these symmetries, then the following necessary conditions are also sufficient for G(p) to be isostatic:*

$e = 3b - 3$ and for any non-empty set of bars B^*, $e^* \leqslant 3b^* - 3$ and

(i) *for $\mathcal{C}_n$, $n > 3$: $b_n = 1$;*
(ii) *for $\mathcal{C}_{2v}$: $b_2 = b_\sigma = 1$ and $e_2 = e_\sigma = 0$ for each mirror;*
(iii) *for $\mathcal{C}_{3v}$: (a) $b_\sigma = 0$ and $e_\sigma = 1$ for each mirror and b_3 is arbitrary or (b) $b_\sigma = 1$ and $e_\sigma = 0$ and b_3 is arbitrary;*
(iv) *for $\mathcal{C}_{nv}$, $n > 3$: $b_n = 1$ and $e_\sigma = 0$ for each mirror.*

An immediate consequence of this theorem and these conjectures is that there is (would be) a polynomial time algorithm to determine whether a given framework in generic position modulo the symmetry group $\mathcal{G}$ is isostatic. Although we do not have a criterion for isostatic bar and joint 'bodies' of symmetry $\mathcal{C}_{nv}$, $n > 3$, this could be handled within this algorithm. Although the Laman type condition of Theorem 1 involves an exponential number of subgraphs of G, there are several algorithms that determine whether it holds in *cbe* steps where *c* is a constant. The pebble game (Hendrickson and Jacobs, 1997) is an example. The additional conditions for being isostatic with the symmetry group $\mathcal{G}$ trivially can be verified in constant time.

5.2. Sufficient conditions for 3D isostatic body-bar frameworks

In 3D, Tay's Theorem becomes:

Theorem 4 (*Tay, 1984*). *For a generic body-bar configuration in 3D, p, the body-bar framework G(p) is isostatic if and only if G = (B,E) satisfies the conditions:*

(i) *$e = 6b - 6$;*
(ii) *for any non-empty set of bodies B^*, which induce just the bars in E^*, with $|B^*| = b^*$ and $|E^*| = e^*$, $e^* \leqslant 6b^* - 6$.*

Equivalently, the body-bar framework G(p) is isostatic in 3-space if and only if G = (B,E) is partitioned into six spanning trees.

If we assume that we start with such a graph, then we can ask whether the additional necessary conditions for a realization that is generic with point group symmetry $\mathcal{G}$ to be isostatic are also sufficient. Without substantial investigation of some of the cases, we provide some sample conjectures.

We note that, in general, the global conditions imply the corresponding conditions for all subgraphs G^* which also have the Tay count $e^* = 6b^* - 6$. For many of these symmetry groups, the condition such as $b_n = 1$ is actually a minimum value of 1 by even simpler counts. For example, with no possible fixed bars for $\mathcal{C}_5$, $b_5 = 0$, both b and e are multiples of 5, and e cannot equal $6(b - 1)$. The extra condition from the group representations is that b_5 cannot be bigger than 1.

The exceptions occur for $\mathcal{C}_2, \mathcal{C}_6$, where the simple Tay counts can occur without the extra added conditions, so we will need to impose an extra subgraph condition. We offer some samples of these conjectures.

Conjecture 2. *If p is a spatial configuration generic with symmetry group $\mathcal{G}$, and G(p) is a framework realized with these symmetries, then the following necessary conditions are also sufficient for G(p) to be isostatic: $e = 6b - 6$ and for any non-empty set of bars B^*, $e^* \leqslant 6b^* - 6$ and*

(i) *for $\mathcal{C}_s$: $e_\sigma = 0$, b_σ is arbitrary;*
(ii) *for $\mathcal{C}_3$: $e_3 = 0$, b_3 is arbitrary;*
(iii) *for $\mathcal{C}_n$ ($n > 3, n \neq 2,6$): $b_n = 1$ and $e_n = 0$;*
(iv) *for $\mathcal{C}_i$: $e_i = 0$ and b_c is arbitrary;*
(v) *for $\mathcal{C}_{3v}$: $e_\sigma = 0$ and b_σ is arbitrary for each mirror; $e_3 = 0$ and b_3 is arbitrary;*
(vi) *for $\mathcal{C}_{nv}$, $n > 3$: $b_n = 1$, $e_n = e_\sigma = 0$ and b_σ is arbitrary for each mirror;*
(vii) *for $\mathcal{C}_{3h}$: $e_3 = e_\sigma = 0$ and b_σ and b_3 are arbitrary;*
(viii) *for $\mathcal{C}_{nh}$, $n > 3$: $b_n = 1$, $e_n = e_\sigma = 0$ and b_σ is arbitrary.*

Here is a sample of the other type of conjectured conditions.

Conjecture 3. *If p is a spatial configuration generic with symmetry group $\mathcal{G}$, and G(p) is a framework realized with these symmetries, then the following necessary conditions are also sufficient for G(p) to be isostatic:*

$e = 6b - 6$ and for any non-empty set of bars B^*, $e^* \leqslant 6b^* - 6$ and

(i) *for $\mathcal{C}_2$: $b_2 = 1$, $e_2 = 0$ or $b_2 = 0$, $e_2 = 2$ and there are no vertex disjoint $\mathcal{C}_2$-symmetric subgraphs G_1^*, G_2^* with $e_1^* = 6b_1^* - 6$ and $e_2^* = 6b_2^* - 6$;*
(ii) *for $\mathcal{C}_6$: $b_6 = 1$ and there are no vertex disjoint $\mathcal{C}_6$-symmetric subgraphs G_1^*, G_2^* with $e_1^* = 6b_1^* - 6$ and $e_2^* = 6b_2^* - 6$.*

As a suggestion that a number of these can be proven, we provide several sufficient conditions which may also be necessary. These are cast in terms of tree coverings which are at the core of various proofs both for Tay's Theorem in all dimensions, and for recent proofs for plane symmetric bar and joint frameworks (Schulze, 2009b, in press-a). A version of this proof places the six spanning trees onto the six edges of a regular tetrahedron (White and Whiteley, 1987). Since this realization has a number of the desired symmetries, we have the following sufficient conditions. Note that these do not, immediately, correspond to the necessary conditions above. There remains significant work to be done.

Theorem 5. *If p is a spatial configuration generic with symmetry group $\mathcal{G}$, and G(p) is a framework realized with these symmetries, then the following conditions are sufficient for G(p) to be isostatic as a body-bar framework:*

(i) *for $\mathcal{C}_s$: we have a partition into six spanning trees $T_1, \ldots, T_6$ with the properties: T_1, T_2 go onto themselves as trees under the mirror, and T_3, T_4 interchange and T_5, T_6 interchange;*
(ii) *for $\mathcal{C}_2$: we have a partition into six spanning trees with the properties: T_1, T_2 go onto themselves as trees under the half-turn, and T_3, T_4 interchange and T_5, T_6 interchange;*
(iii) *for $\mathcal{C}_3$: we have a partition into six spanning trees with the properties: T_1, T_2, T_3 cycle as trees under the turn, and T_4, T_5, T_6 cycle as trees under the turn.*

6. Extensions and further work

6.1. Identified attachment points

As we noted in the introduction, with the second version of the Stewart Platform (Fig. 1b), in applications it is common to have some end-points or attachment points of bars coinciding on a body. What analysis extends to those situations?

In the plane, this is not an issue, as we also have a complete set of necessary conditions, and some complete sufficient conditions, for bar and joint frameworks, and the body-bar frameworks can be embedded into that theory, with the exception of finding initial bodies with full symmetry $\mathcal{C}_n$, $(n > 3)$.

In 3D, the necessary conditions for symmetric body-bar frameworks to be isostatic established in this paper also extend to body-bar frameworks that have some of their attachment points on the bodies identified. In fact, just like in the 2-dimensional case, the necessity of these conditions for either type of body-bar structure can be verified by translating the results on bar and joint frameworks derived in Connelly et al. (2009). Note, however, that for 3D body-bar frameworks with identified end-points, there could be additional necessary conditions (such as conditions on the number of end-points on the bodies, for example).

The problem of establishing *sufficient* conditions for body-bar frameworks with identified end-points in 3D is complex: we do not have a general form of Tay's Theorem with end-points of bars identified. On the other hand, the connection to laying six trees onto a tetrahedron, where three trees coincide at each end-point, does indicate that a number of coinciding end-points are possible, and this also extends to some realizations with some symmetries. Given the potential applications this is a significant topic for further investigation.

6.2. Body-hinge structures

Another structural type of interest are body-hinge structures, in which bodies are connected by revolute hinges along assigned lines. These hinges function as implicit packages of five bars meeting the assigned hinge line. For generic hinges, there is a version of Tay's Theorem, without symmetry (Whiteley, 1988). Therefore we anticipate that there are symmetry extensions for this situation (which implicitly includes some identifications of bars). This is currently work in progress. This extension is a necessary step towards applying these results directly to the rigidity and flexibility of biomolecules (Whiteley, 2005).

6.3. Modeling body-bar frameworks as bar and joint frameworks

In our definition of a symmetry operation of a body-bar framework, we used the extended graph $\overline{G}$ which models each body of G as the complete graph on the vertices of the attached bars. We used this definition of a symmetry operation through the rest of the paper, but did not make any use of the rigidity properties of the frameworks on the bodies – beyond assuming that each body was rigid in Tay's Theorem and in Eqs. (2)–(4).

If we want to translate the results of this paper to bar and joint frameworks $\widetilde{G}(p)$ modeling the body-bar frameworks, we can substitute an arbitrary isostatic framework for each body. With this substitution, necessary conditions for isostatic body-bar frameworks extend to necessary conditions for the corresponding isostatic bar and joint frameworks $\widetilde{G}(p)$. Notice that it is not necessary that the symmetries of a body (which are symmetries of the attachment points on the body) are actually automorphisms of the substituted framework.

For example, for C_4 in the plane, an isostatic body-bar framework has an unshifted body – but there is no isostatic bar and joint framework in the plane which has C_4 symmetry as a graph automorphism (Connelly et al., 2009).

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